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Vector Fields and Their Duals

Philip Feinsilver

Department of Mathematics, Southern Illinois University, Carbondale, Illinois 62901

and

René Schott

Institut E. Cartan and Loria, Université Henri Poincaré-Nancy 1, 54506 Vandoeuvre-lès-Nancy, France

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A standard way of realizing a Lie algebra is as a family of vector fields closed under commutation. Using the action on the universal enveloping algebra, one finds a realization in a dual form—the double dual. This is an algebraic Fourier transform of a "vector fields realization" of the Lie algebra. On the other hand, in the subject of umbral calculus (canonical boson calculus) the duals-to-vector fields play a primary role. It is shown that the double dual realizations of Lie algebras provide a rich source of examples for the umbral calculus, which, complementarily, provides a canonical construction of polynomial systems associated to the Lie algebra. For any finite-dimensional Lie algebra, take an element in the local Lie group it generates. Then there is an abelian family of operators such that acting on a canonical vacuum state, the abelian group gives the same result as the group element constructed via the given Lie algebra. In other words, they yield the same coherent states.

INTRODUCTION

We review the basic constructions involved first in the umbral calculus context, specifically the holomorphic canonical calculus. Then we recall the double dual construction associated to a finite-dimensional Lie algebra with a chosen basis. The connection with umbral calculus is shown. An example is included to illustrate the correspondence between the two points of view.

Notation. We use a summation convention somewhat more general than that of the Einstein convention, namely, Greek indices are always summed, regardless of position. Latin indices are summed only if the summation is indicated explicitly.

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2. HOLOMORPHIC CANONICAL CALCULUS

Start with a function $V: \mathbf{C}^N \to \mathbf{C}^N$, $V(z) = (V_1(z_1, ..., z_N), ..., V_N(z_1, ..., z_N))$ holomorphic in a neighborhood of the origin, satisfying $V(0) = 0$. The variables z_i, x_j are *canonical duals*, i.e., they satisfy the canonical commutation relations, CCR, $[z_i, x_j] = \delta_{ij}$, $[z_i, z_j] = [x_i, x_j] = 0$. (All physical units are scaled out.) We think of z_i as the partial derivative operator $\partial_i = \partial/\partial x_i$ acting on germs of holomorphic functions of $(x_1, ..., x_N)$. Another terminology is that they are standard boson operators: creation (modelled as multiplication by x_i) and annihilation (modelled as partial differentiation z_i) operators. In this context, a function of $x = (x_1, ..., x_N)$, $f(x)$, is identified with $f(x)\,1$, the operator of multiplication by $f(x)$ acting on the *vacuum state* 1, with $z_i 1 = 0$, for all $1 \leqslant i \leqslant N$.

Denoting the Jacobian $V'(z)$, let $W(z) = (V'(z))^{-1}$, be the inverse (matrix inverse) Jacobian. Then the CCR extend to $[V(z), x_i] = \partial V/\partial z_i$ and thus, setting the operators

$$\hat{Y}_i = x_\mu W_{\mu i}(z)$$

yields the system of bosons $\{V_i, \hat{Y}_j\}_{1 \leqslant i, j \leqslant N}$, with $[V_i, \hat{Y}_j] = \delta_{ij}$. The essential feature is that, indeed, $[\hat{Y}_i, \hat{Y}_j] = [V_i, V_j] = 0$. Notice that exchanging z with x is an algebraic Fourier transformation and turns the variables $\hat{Y}_i$ into the *vector fields* $Y_i = W(x)_{\mu i}\,\partial_\mu$. Thus, the $\hat{Y}_i$ are "dual vector fields."

Notation. To complete the standard notations used along with V and W, let U denote the inverse function to V; i.e., $U \circ V = V \circ U = \mathrm{id}$. Explicitly, $U(V(z)) = z$.

Note that we use a dot to denote differentiation with respect to t, where t is a real parameter.

As operators on functions of x, we identify $z = (z_1, ..., z_N) = \partial = (\partial_1, ..., \partial_N)$, operators of partial differentiation with respect to the x_i.

To formalize our discussion we introduce

DEFINITION 2. A *dual vector field*, dvf for short, is an operator dual to a vector field; i.e., an operator of the form

$$\hat{X} = x_\mu F_\mu(\partial),$$

where $F: \mathbf{C}^N \to \mathbf{C}$ is holomorphic in a neighborhood of the origin in $\mathbf{C}^N$.

In this case, $\hat{X}$ is dual to the vector field $X = F_\mu(x)(\partial/\partial x_\mu)$.

Remark 2.2. See [1, pp. 17–27, Chap. 1] for details on the holomorphic canonical calculus.

2.1. *Flow Generated by Canonical Variables*

By a *Hamiltonian* is meant an operator generating a time-evolution. Take as Hamiltonian the linear combination

$$\hat{H}=\alpha_\lambda \hat{Y}_\lambda=\alpha_\lambda x_\mu W_{\mu\lambda}(z).$$

Compare this with the vector field

$$H=\alpha_\lambda W_{\mu\lambda}(x)\frac{\partial}{\partial x_\mu}.$$

We consider the phase-space flows these generate, focusing attention on the x-variables. We solve Hamilton's equations,

$$\frac{d}{dt}x_i(t)=\dot{x}_i=\frac{\partial H}{\partial z_i}, \qquad \frac{d}{dt}z_i(t)=\dot{z}_i=-\frac{\partial H}{\partial x_i},$$

$1\leqslant i\leqslant N$, equivalently, employ the method of characteristics, to find our results, sometimes working mainly in the x-space, sometimes mainly in the z-space.

Proposition 2.3. *Let $H=\alpha_\lambda W_{\mu\lambda}(x)(\partial/\partial x_\mu)$. Then the x_i variables evolve according to*

$$x(t)=e^{tH}xe^{-tH}=U(V(x)+\alpha t)$$

equivalently, $V(x(t))=V(x)+\alpha t$.

Proof. We have to solve $\dot{x}_i=\alpha_\lambda W_{i\lambda}(x)$. Multiplying by V', we have an exact derivative,

$$\frac{d}{dt}V(x(t))=\alpha.$$

Integrating, with the initial condition $x(0)=x$, yields the result. ∎

Note that this is "trivial" in the sense that the flow is linear and it is a change-of-coordinates that is going on. On the other hand, the flow generated by the dvf $\hat{H}$ plays an essential role in umbral calculus. Here we use the variables z for the partial derivative operators.

Proposition 2.4. *Let $\hat{H}=\alpha_\lambda x_\mu W_{\mu\lambda}(\partial)$. Then, with e^{ax} denoting* $\exp(a_\mu x_\mu)$, *we have the action*

$$e^{t\hat{H}}e^{ax}=\exp(xU(V(a)+\alpha t)).$$

Note. In the boson calculus, this is the same as

$$e^{t\hat{H}}e^{ax}1 = \exp(xU(V(a)+\alpha t))\,1.$$

Proof. We use the tactic of going over to the flow in the z-variables. Then, interpreting z's as partial derivatives, we find a differential equation for $e^{t\hat{H}}e^{ax}$. Start with $\dot{z}_i = -\alpha_\lambda W_{i\lambda}(z)$.

As in Proposition 2.3, multiplying by V' yields an exact derivative, and $z(t) = U(V(z) - \alpha t)$. Apply this to e^{ax}. Now, $ze^{ax} = ae^{ax}$, and we find, changing $t \to -t$,

$$\begin{aligned} e^{-t\hat{H}}ze^{t\hat{H}}e^{ax}1 &= U(V(z)+\alpha t)\,e^{ax}1 \\ &= U(V(a)+\alpha t)\,e^{ax}1 \end{aligned}$$

and applying $e^{t\hat{H}}$ to both sides yields

$$ze^{t\hat{H}}e^{ax}1 = U(V(a)+\alpha t)\,e^{t\hat{H}}e^{ax}1.$$

Thus, interpreting the action of z as differentiation, with the initial condition $e^{t\hat{H}}e^{ax}|_{x=0} = 1$,

$$e^{t\hat{H}}e^{ax} = \exp(x_\mu U_\mu(V(a)+\alpha t)). \quad \blacksquare$$

Recalling the above construction of the $\hat{Y}$ variables, we see that, setting $a = 0$, $t = 1$, using standard multi-index notations,

$$e^{\alpha\hat{Y}}1 = e^{\hat{H}}1 = e^{xU(\alpha)} = \sum_{n\geqslant 0} \frac{\alpha^n}{n!}\,\eta_n(x), \tag{1}$$

where $\eta_n(x) = \hat{Y}^n 1$ are the *canonical basis polynomials*, multivariate *polynomials of binomial type*.

Remark 2.5. The approach of H. Gzyl [2, 3] via integral operators is interesting in this context.

Now from the Lie side.

3. DOUBLE DUAL REPRESENTATIONS

Start with a finite-dimensional Lie algebra with basis $\{\xi_1, ..., \xi_N\}$. Define (local) group elements

$$g(A, \xi) = e^{A_1\xi_1} \cdots e^{A_N\xi_N}.$$

The fundamental theorems of Lie imply that there is a group law

$$g(A, \xi)\, g(B, \xi) = g(A \odot B, \xi),$$

where $A \odot B$ is analytic in the variables $A = (A_1, ..., A_N)$, $B = (B_1, ..., B_N)$. Note that $g(0, \xi)$ is the identity.

Further, Lie's fundamental theorems, or Frobenius integrability theory, imply that there is a realization of the ξ_i as vector fields $\xi_i^{\ddagger}$ acting on the group elements $g(A, \xi)$ on the left. Namely,

$$\xi_i g(A, \xi) = \xi_i^{\ddagger} g(A, \xi) = (\pi_{i\mu}^{\ddagger}(A)\, \partial_\mu)\, g(A, \xi)$$

here with $\partial_i = \partial/\partial A_i$.

The idea is to expand g in powers of the A variables to see that g is effectively the generating function of the Poincaré–Birkhoff–Witt (PBW) basis of the universal enveloping algebra,

$$g(A, \xi) = \sum_{n \geqslant 0} \frac{A^n}{n!} \xi_1^{n_1} \cdots \xi_N^{n_N}$$

so that left-multiplication by ξ_i is dual to the action of the vector field $\xi_i^{\ddagger}$ in the sense that the action of ξ on the PBW basis is transferred via the generating function g to a differential operator acting on functions of the A-variables.

Now, there is another way to interpret this action that leads to the double dual. Introduce canonical bosons acting on the (PBW) basis thus

$$\mathscr{R}_i \xi_1^{n_1} \cdots \xi_i^{n_i} \cdots \xi_N^{n_N} = \xi_1^{n_1} \cdots \xi_i^{n_i+1} \cdots \xi_N^{n_N}$$

$$\mathscr{V}_i \xi_1^{n_1} \cdots \xi_i^{n_i} \cdots \xi_N^{n_N} = n_i \xi_1^{n_1} \cdots \xi_i^{n_i-1} \cdots \xi_N^{n_N},$$

formal raising (creation) and lowering (annihilation) operators. Then the action of a vector field $\xi_i^{\ddagger} = \pi_{i\mu}^{\ddagger}(A)\, \partial_\mu$ dualizes to the dvf $\hat{\xi}_i = \mathscr{R}_\mu \pi_{i\mu}^{\ddagger}(\mathscr{V})$. This is the *double dual* representation. It is a boson realization of the action of left multiplication of the basis elements ξ_i on the PBW basis.

Remark 3.1. For the pi-matrices, see [1, pp. 28–30, Chap. 2]. More on the double dual is found in [1, pp. 35–36/40].

Next, convert from $\mathscr{R}, \mathscr{V}$ to x, z variables. So, from the construction immediately follows this important observation.

PROPOSITION 3.2. *Let* $\hat{\xi}$ *be the double dual in variables* (x, z), *so* $\hat{\xi}_i = x_\mu \pi^\ddagger_{i\mu}(z)$. *Then, for any group elements or polynomial functions* $F(\xi)$, $G(\xi)$,

$$F(\hat{\xi})\, G(x) = F(\xi)\, G(\xi)|_{\xi \to x}.$$

Let $X = \alpha_\mu \xi_\mu$ be a typical element of the Lie algebra. Define the coordinate mapping $\alpha \to A$ by

$$\exp(X) = \exp(\alpha_\mu \xi_\mu) = e^{A_1(\alpha)\,\xi_1} \cdots e^{A_N(\alpha)\,\xi_N} = g(A(\alpha), \xi).$$

PROPOSITION 3.3. *With* $X^\ddagger = \alpha_\mu \xi^\ddagger_\mu$, *we have the one-parameter group* $\exp(tX^\ddagger)$ *and*

$$\exp(tX^\ddagger)\, g(A, \xi) = g(A(\alpha t) \odot A, \xi).$$

Proof. One application of X to g is the same as applying $X^\ddagger$, i.e., $Xg(A, \xi) = X^\ddagger g(A, \xi)$ by definition of the left dual. Now, $XX^\ddagger g = X^\ddagger X g$ since $X^\ddagger$ and X act independently, $X^\ddagger$ on A-variables, X on the ξ variables. Iterating applications of X and $X^\ddagger$ builds up the exponential. ∎

Note that the characteristic equations for the flow generated by $X^\ddagger = \alpha_\lambda \pi^\ddagger_{\lambda\mu}(A)\, \partial_\mu$ are

$$\dot{A}_i = \alpha_\lambda \pi^\ddagger_{\lambda i}(A) \tag{2}$$

which we see has the solution $A(t) = A(\alpha t) \odot A$.

Setting $t = 1$ yields

$$e^{X^\ddagger} g(A, \xi) = g(A(\alpha) \odot A, \xi).$$

From Proposition 3.2, we have the action of the double dual

$$\begin{aligned} e^{\hat{X}} g(A, x) &= e^X g(A, \xi)|_{\xi \to x} \\ &= g(A(\alpha), \xi)\, g(A, \xi)|_{\xi \to x} \\ &= g(A(\alpha) \odot A, x). \end{aligned}$$

Replacing A by a in $g(A, x)$, we thus have

$$e^{\hat{X}} e^{ax} = e^{(A(\alpha) \odot a)\, x}. \tag{3}$$

Compare with

$$\exp(\alpha_\mu \hat{Y}_\mu)\, e^{ax} = \exp(x_\mu U_\mu(V(a) + \alpha)) \tag{4}$$

from Proposition 2.4.

Thus, the main

THEOREM 3.4. *Acting on the vacuum state 1, the group elements generated by the double dual $\hat{X}$ and the canonical variable $\alpha_\mu \hat{Y}_\mu$ give the same result*

$$e^{\hat{X}}1 = \exp(xA(\alpha)) = e^{\alpha_\mu \hat{Y}_\mu}1 = \exp(xU(\alpha))$$

under the correspondence of the momentum variables with the coordinates

$$z \leftrightarrow A, \qquad V \leftrightarrow \alpha,$$

i.e., the canonical operators $\hat{Y}_i$ are given as $x_\mu W_{\mu i}(\partial)$ where W is the inverse Jacobian matrix of the coordinate map $A \to \alpha$, equivalently, the Jacobian matrix of the coordinate map $\alpha \to A$ expressed in the A variables, then replacing every A_i by the corresponding partial differentiation operator ∂_i.

Proof. This follows from the above discussion by setting $a = 0$ in Eqs. (3) and (4). ∎

Referring to Eq. (1), we thus have associated a system of polynomials $\{\eta_n\}$

$$e^{xA(\alpha)} = \sum_{n \geqslant 0} \frac{\alpha^n}{n!} \eta_n(x)$$

to any Lie algebra with a specified basis.

4. CANONICAL VARIABLES IN THE NONABELIAN CASE

Now we can combine the two: canonical variables in the Lie case. Following the approach of Proposition 2.4,

PROPOSITION 4.1. *Let $\hat{H} = x_\nu \alpha_\mu \pi^{\ddagger}_{\mu\lambda}(V(\partial))\, W_{\nu\lambda}(\partial)$. Then*

$$e^{t\hat{H}} e^{ax} = \exp(xU(A(\alpha t) \odot V(a))).$$

Proof. We have

$$\dot{z}_i = -\alpha_\mu \pi^{\ddagger}_{\mu\lambda}(V(z))\, W_{i\lambda}(z).$$

Multiplying by V', we have the characteristic Eqs. (2) in the variables $V(z)$. So

$$V(z(t)) = A(-\alpha t) \odot V(z).$$

And, switching $t \to -t$,

$$e^{-t\hat{H}} z e^{t\hat{H}} e^{ax} 1 = U(A(\alpha t) \odot V(z))\, e^{ax} 1$$
$$= U(A(\alpha t) \odot V(a))\, e^{ax} 1.$$

Applying $e^{t\hat{H}}$ to both sides now yields

$$z e^{t\hat{H}} e^{ax} 1 = U(A(\alpha t) \odot V(a))\, e^{t\hat{H}} e^{ax} 1$$

and the result follows as in the proof of Proposition 2.4. ▌

And

THEOREM 4.2. *To the vector fields*

$$\xi_i^{\ddagger} = \pi_{i\lambda}^{\ddagger}(V(x))\, W_{\nu\lambda}(x)\, \partial_\nu$$

correspond the dvf's

$$\hat{\xi}_i = x_\nu \pi_{i\lambda}^{\ddagger}(V(\partial))\, W_{\nu\alpha}(\partial),$$

and with $\hat{X} = \alpha_\mu \hat{\xi}_\mu$,

$$e^{\hat{X}} 1 = e^{xU(A(\alpha))}.$$

Now choose U and A to be inverse maps, i.e., $V(z) = A(z)$. Then we have the nonabelian Lie algebra yielding the same result on the vacuum state, 1, as the abelian one, namely

$$\exp(\hat{X})\, 1 = \exp(\alpha x).$$

5. CONSTRUCTS SUMMARY

Here is an outline of the elements of the theory.

1. Initial data consisting of Lie algebra with given basis.

2. From the left dual, $\pi^{\ddagger}$, comes the double dual realization.

3. From the equations $\dot{A} = \alpha \pi^{\ddagger}(A)$ with initial conditions $A(0) = A$, find $A(\alpha t) \odot A$, and hence the map $\alpha \to A$, evaluating at $A = 0$, $t = 1$.

4. Interpretation of A as momentum variables, α as canonical momenta. Dual variables x to A, $\hat{Y}$ to α.

5. Jacobians. $\partial A / \partial \alpha$, expressed in terms of A, used for the raising operators $\hat{Y}$. $\partial \alpha / \partial A$ in terms of α computed as the algebraic inverse

$(\partial A/\partial\alpha)^{-1}$, used to express the variables x in terms of raising and lowering operators.

6. Generic formulae $\hat{Y} = xV'(z)^{-1} = xU'(V(z))$, $x = \hat{Y}V'(z) = \hat{Y}U'(V)^{-1}$ become

$$\hat{Y} = xA'(\alpha(A)), \qquad x = \hat{Y}A'(\alpha)^{-1}.$$

7. Canonical polynomials $\eta_x(x) = \hat{Y}^n 1$. Raising/lowering operators on the basis η_n

$$\hat{Y}_i \eta_n = \eta_{n+e_i}$$

$$\mathscr{V}_i \eta_n = n_i \eta_{n-e_i}.$$

Acting on the basis η_n, x's yield *recursion formulas*. Basic expressions (row vector times matrix),

$$\hat{Y} = xA'(\alpha(\partial))$$

$$x = \hat{Y}A'(\mathscr{V})^{-1}$$

with $\partial = (\partial_1, ..., \partial_N)$ partial differentiation with respect to x-variables,

8. Including the change-of-variables in $\hat{\xi}$ yields the general

$$\hat{\xi}_i = x_\nu W_{\nu\lambda}(\partial)\, \pi^\ddagger_{i\lambda}(V(\partial))$$

with

$$e^{\alpha_\mu \hat{\xi}_\mu} 1 = e^{x_\mu U_\mu(A(\alpha))}.$$

9. In particular

$$\hat{\xi} = x_\nu A'(\partial)^{-1}_{\nu\lambda} \pi^\ddagger_{i\lambda}(A(\partial))$$

yields

$$e^{\alpha_\mu \hat{\xi}_\mu} 1 = e^{\alpha_\mu x_\mu}.$$

6. ILLUSTRATION: HEISENBERG ALGEBRA

To illustrate, take the 3-dimensional Heisenberg algebra, with non-trivial commutation relation $[\xi_3, \xi_1] = \xi_2$. For the left dual [1, pp. 176–178],

$$\pi^\ddagger(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & A_1 & 1 \end{pmatrix} \tag{5}$$

with the double dual

$$\hat{\xi}_1 = \mathscr{R}_1, \qquad \hat{\xi}_2 = \mathscr{R}_2, \qquad \hat{\xi}_3 = \mathscr{R}_3 + \mathscr{R}_2 \mathscr{V}_1.$$

Solving $\dot{A} = \alpha\pi^{\ddagger}(A)$, using Eq. (5), with the initial conditions $A(0) = A$, yields

$$\begin{aligned} A_1(t) &= A_1 + \alpha_1 t \\ A_2(t) &= A_2 + \alpha_2 t + \alpha_3 A_1 t + \alpha_1 \alpha_3 t^2/2 \\ A_3(t) &= A_3 + \alpha_3 t. \end{aligned}$$

With $t = 1$, this is $A(\alpha) \odot A$, and with $A = 0$ this gives the coordinate map $\alpha \to A$,

$$\begin{aligned} A_1(\alpha) &= \alpha_1 \\ A_2(\alpha) &= \alpha_2 + \alpha_1 \alpha_3/2 \\ A_3(\alpha) &= \alpha_3. \end{aligned} \tag{6}$$

The Jacobians are

$$\frac{\partial A}{\partial \alpha} = \begin{pmatrix} 1 & 0 & 0 \\ \alpha_3/2 & 1 & \alpha_1/2 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \left(\frac{\partial A}{\partial \alpha}\right)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -\alpha_3/2 & 1 & -\alpha_1/2 \\ 0 & 0 & 1 \end{pmatrix} \tag{7}$$

with the latter being $\partial\alpha/\partial A$ in terms of α. We see from Eqs. (6) that, in terms of A,

$$\frac{\partial A}{\partial \alpha}(A) = \begin{pmatrix} 1 & 0 & 0 \\ A_3/2 & 1 & A_1/2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Contracting with x and replacing A by ∂ yields the raising operators (canonical variables)

$$\hat{Y}_1 = x_1 + \tfrac{1}{2} x_2 \partial_3, \qquad \hat{Y}_2 = x_2, \qquad \hat{Y}_3 = x_3 + \tfrac{1}{2} x_2 \partial_1.$$

These are commuting variables. The basic expansion is

$$e^{\alpha_\mu \hat{Y}_\mu} 1 = e^{\alpha_1 x_1} e^{x_2(\alpha_2 + \alpha_1\alpha_3/2)} e^{\alpha_3 x_3} = \sum_{n \geqslant 0} \frac{\alpha^n}{n!} \eta_n(x).$$

Contracting with $\hat{Y}$ and replacing α by $\mathscr{V}$ in the second equation (7) yields the x-variables in terms of raising/lowering operators,

$$x_1 = \hat{Y}_1 - \hat{Y}_2 \mathscr{V}_3/2, \qquad x_2 = \hat{Y}_2, \qquad x_3 = -\hat{Y}_2 \mathscr{V}_1/2 + \hat{Y}_3. \tag{8}$$

On the basis η_n, the action of the $\hat{Y}_i$ is given by $\hat{Y}_i \eta_n = \eta_{n+e_i}$ while the x_i yield recursion relations,

$$\begin{aligned} x_1 \eta_n &= \eta_{n+e_1} - \tfrac{1}{2} n_3 \eta_{n+e_2-e_3} \\ x_2 \eta_n &= \eta_{n+e_2} \\ x_3 \eta_n &= \eta_{n+e_3} - \tfrac{1}{2} n_1 \eta_{n-e_1+e_2}. \end{aligned}$$

Finally, in Eqs. (8), replacing $\hat{Y}$ by x, $\mathscr{V}$ by ∂, and contracting with the transpose of $\pi^{\ddagger}(A(\partial))$, as in Part 9 of the constructs summary, yield

$$\hat{\xi}_1 = x_1 - \tfrac{1}{2} x_2 \partial_3, \qquad \hat{\xi}_2 = x_2, \qquad \hat{\xi}_3 = x_3 + \tfrac{1}{2} x_2 \partial_1$$

which satisfy the commutation relations for the Heisenberg algebra while satisfying $\exp(\alpha_\mu \hat{\xi}_\mu)\, 1 = \exp \alpha_\mu x_\mu$.

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